



On the Hausdorff and packing measures of typical compact metric spaces

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Abstract. We study the Hausdorff and packing measures of typical compact metric spaces belonging to the Gromov–Hausdorff space (of all compact metric spaces) equipped with the Gromov–Hausdorff metric.

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1. Introduction

Recall that a subset E of a metric space M is called co-meagre if its complement is meagre; also recall that if P is a property that the elements of M may have, then we say that a typical element x in M has property P if the set $E = \{x \in M \mid x \text{ has property } P\}$ is co-meagre, see Oxtoby [9] for more details. The purpose of this paper is to investigate the Hausdorff and packing measures of a typical compact metric space belonging to the Gromov–Hausdorff space K_{GH} of all compact metric spaces; the precise definition of the Gromov–Hausdorff space K_{GH} will be given below. The four most commonly used fractal dimensions of a metric space X are: the lower and upper box dimensions, denoted by $\underline{\dim}_B(X)$ and $\overline{\dim}_B(X)$, respectively; the Hausdorff dimension, denoted by $\dim_H(X)$; and the packing dimension, denoted by $\dim_P(X)$; the precise definitions will be given in Sect. 2.2. It is well-known that if X is a metric space, then these dimensions satisfy the following inequalities,

$$\begin{aligned}\dim_H(X) &\leq \dim_P(X) \leq \overline{\dim}_B(X), \\ \dim_H(X) &\leq \underline{\dim}_B(X) \leq \overline{\dim}_B(X).\end{aligned}$$

We now return to the main question in this paper: what are the dimensions of a typical compact metric space? Rouyer [13] has very recently provided the following answer to this question.

Theorem A. [13] *A typical compact metric space $X \in K_{\text{GH}}$ satisfies*

$$\dim_{\text{H}}(X) = \underline{\dim}_{\text{B}}(X) = 0, \\ \overline{\dim}_{\text{B}}(X) = \infty.$$

Theorem A shows that the lower box dimension of a typical compact metric space is as small as possible and that the upper box dimension of a typical compact metric space is as big as possible. Other studies of typical compact sets show the same dichotomy. For example, Gruber [3] and Myjak & Rudnicki [8] proved that if X is a metric space, then the lower box dimension of a typical compact subset of X is as small as possible and that the upper box dimension of a typical compact subset of X is (in many cases) as big as possible. The purpose of this paper is to analyse this intriguing dichotomy, and, in particular, the dichotomy in Theorem A, in more detail.

For example, as an application of our main results we show that not only is the upper box dimension of a typical compact metric space $X \in K_{\text{GH}}$ equal to infinity (see Theorem A above), but even the smaller packing dimension is equal to infinity; this is the content of Theorem 1.1 below.

Theorem 1.1. *A typical compact metric space $X \in K_{\text{GH}}$ satisfies*

$$\dim_{\text{P}}(X) = \infty.$$

While Theorems A and 1.1 study and compute the dimensions of typical compact metric spaces, we prove more general results investigating and computing not only the dimensions of typical compact metric spaces but also the exact values of the Hausdorff and packing measures of typical compact metric spaces, see Theorem 2.4.

In fact, we prove even stronger results providing information about the so-called Hewitt–Stromberg measures of typical compact spaces, see Theorems 2.2 and 2.3; the results in Theorem 2.4 on the exact values of the Hausdorff and packing measures of typical compact metric spaces follow immediately from Theorems 2.2 and 2.3.

The paper is structured as follows.

We first recall the definition of the Gromov–Hausdorff space and the Gromov–Hausdorff metric in Sect. 2.1.

In Sects. 2.2–2.3 we recall the definitions of the fractal dimensions and measures investigated in the paper. The definitions of the Hausdorff and packing measures (and the Hausdorff and packing dimensions) are recalled in Sect. 2.2 and the definitions of the Hewitt–Stromberg measures are recalled in Sect. 2.3.

Sections 2.4–2.6 contain our main results. In Sect. 2.4 we investigate and compute the exact values of the Hewitt–Stromberg measures of typical compact metric spaces. Sections 2.5–2.6 contain several applications and corollaries of the results in Sect. 2.4: in Sect. 2.5 we apply the results from Sect. 2.4 to find the exact values of the Hausdorff and packing measures of typical compact spaces, and in Sect. 2.6 we specialise even further and apply the results from Sect. 2.4 to find exact values of the packing dimension (and box dimensions) of typical compact metric spaces.

Finally, the proofs are given in Sects. 3–6.

2. Statements of results

2.1. The Gromov–Hausdorff space K_{GH} and the Gromov–Hausdorff metric d_{GH}

We define the pre-Gromov–Hausdorff space \mathcal{K}_{GH} by

$$\mathcal{K}_{\text{GH}} = \left\{ X \mid X \text{ is a compact and non-empty metric space} \right\}.$$

Next, we define the equivalence relation \sim in \mathcal{K}_{GH} as follows, namely, for $X, Y \in \mathcal{K}_{\text{GH}}$, write

$$X \sim Y \iff \text{there is a bijective isometry } f : X \rightarrow Y.$$

It is clear that \sim is an equivalence relation \sim in \mathcal{K}_{GH} , and the Gromov–Hausdorff space K_{GH} is now defined as the space of equivalence classes, i.e.

$$K_{\text{GH}} = \mathcal{K}_{\text{GH}} / \sim.$$

While the elements of K_{GH} are equivalence classes of compact metric spaces, we will use the standard convention and identify an equivalence class with its representative, i.e. we will regard the elements of K_{GH} as compact metric spaces and not as equivalence classes of compact metric spaces. Next, we define the Gromov–Hausdorff metric d_{GH} on K_{GH} . If Z is a metric space, and A and B are compact subsets of Z , then the Hausdorff distance $d_{\text{H}}(A, B)$ between A and B is defined by

$$d_{\text{H}}(A, B) = \max \left(\sup_{x \in A} \text{dist}(x, B), \sup_{y \in B} \text{dist}(y, A) \right),$$

where $\text{dist}(z, E) = \inf_{x \in E} d(z, x)$ for $z \in Z$ and $E \subseteq Z$. The Gromov–Hausdorff metric d_{GH} on K_{GH} is now defined by

$$d_{\text{GH}}(X, Y) = \inf \left\{ d_{\text{H}}(f(X), g(Y)) \mid \begin{array}{l} Z \text{ is a complete metric space} \\ \text{and } f : X \rightarrow Z \text{ and } g : Y \rightarrow Z \text{ are isometries} \end{array} \right\},$$

for $X, Y \in K_{\text{GH}}$. It is well-known that $(K_{\text{GH}}, d_{\text{GH}})$ is a complete metric space [11]. The reader is referred to [11, Chapter 10], for a detailed discussion of the Gromov–Hausdorff space and the Gromov–Hausdorff metric.

2.2. Hausdorff measure, packing measure and box dimensions

While the definitions of the Hausdorff and packing measures (and the Hausdorff and packing dimensions) and box dimensions are well-known, we have, nevertheless, decided to briefly recall the definitions below. There are several reasons for this: firstly, since we are working in general (compact) metric spaces, the different definitions that appear in the literature may not all agree and for this reason it is useful to state precisely the definitions that we are using; secondly, and perhaps more importantly, the less well-known Hewitt–Stromberg measures (which will be defined below in Sect. 2.3) play an important part in this paper and to make it easier for the reader to compare and contrast the definitions of the Hewitt–Stromberg measures and the definitions of the Hausdorff and packing measures it is useful to recall the definitions of the latter measures; and thirdly, in order to provide a motivation for the Hewitt–Stromberg measures.

Let X be a metric space and let d be the metric in X . For $x \in X$ and $r > 0$, let $C(x, r)$ denote the closed ball with centre at x and radius equal to r , i.e. $C(x, r) = \{y \in X \mid d(x, y) \leq r\}$. The lower and upper box dimensions of a subset E of X are defined as follows. For $r > 0$, the covering number $N_r(E)$ and the packing number $M_r(E)$ of E are defined by

$$\begin{aligned} N_r(E) &= \inf \left\{ |I| \mid (C(x_i, r))_{i \in I} \text{ is a family of closed balls} \right. \\ &\quad \left. \text{with } x_i \in E \text{ and } E \subseteq \cup_i C(x_i, r) \right\}, \\ M_r(E) &= \sup \left\{ |I| \mid (C(x_i, r))_{i \in I} \text{ is a family of closed balls} \right. \\ &\quad \left. \text{with } x_i \in E \text{ and } d(x_i, x_j) \geq r \text{ for } i \neq j \right\}. \end{aligned} \quad (2.1)$$

The lower and upper box dimensions, denoted by $\underline{\dim}_{\text{B}}(E)$ and $\overline{\dim}_{\text{B}}(E)$, respectively, are now defined by

$$\begin{aligned} \underline{\dim}_{\text{B}}(E) &= \liminf_{r \searrow 0} \frac{\log N_r(E)}{-\log r} = \liminf_{r \searrow 0} \frac{\log M_r(E)}{-\log r}, \\ \overline{\dim}_{\text{B}}(E) &= \limsup_{r \searrow 0} \frac{\log N_r(E)}{-\log r} = \limsup_{r \searrow 0} \frac{\log M_r(E)}{-\log r}; \end{aligned} \quad (2.2)$$

the fact that the lower limits $\liminf_{r \searrow 0} \frac{\log N_r(E)}{-\log r}$ and $\liminf_{r \searrow 0} \frac{\log M_r(E)}{-\log r}$ coincide and the fact that the upper limits $\limsup_{r \searrow 0} \frac{\log N_r(E)}{-\log r}$ and $\limsup_{r \searrow 0} \frac{\log M_r(E)}{-\log r}$ coincide is proven in [3].

Next, we recall the definitions of the Hausdorff and packing measures. We start by recalling the definition of a dimension function.

Definition. (Dimension function) *A function $h : (0, \infty) \rightarrow (0, \infty)$ is called a dimension function if h is increasing, right continuous and $\lim_{r \searrow 0} h(r) = 0$.*

The Hausdorff measure associated with a dimension function h is defined as follows. Let X be a metric space and $E \subseteq X$. For $\delta > 0$, we write

$$\mathcal{H}_\delta^h(E) = \inf \left\{ \sum_i h(\text{diam}(E_i)) \mid E \subseteq \bigcup_{i=1}^{\infty} E_i, \text{diam}(E_i) < \delta \right\}.$$

The h -dimensional Hausdorff measure $\mathcal{H}^h(E)$ of E is now defined by

$$\mathcal{H}^h(E) = \sup_{\delta > 0} \mathcal{H}_\delta^h(E).$$

If $t > 0$ and h_t denotes the dimension function defined by $h_t(r) = r^t$, then we will follow the traditional convention and write

$$\mathcal{H}^{h_t}(E) = \mathcal{H}^t(E).$$

Finally, the Hausdorff dimension $\dim_{\mathcal{H}}(E)$ is defined by

$$\dim_{\mathcal{H}}(E) = \sup \{t \geq 0 \mid \mathcal{H}^t(E) = \infty\}.$$

The reader is referred to Rogers' classical text [12] for an excellent and systematic discussion of the Hausdorff measures \mathcal{H}^h .

The packing measure with a dimension function h is defined as follows. For $E \subseteq X$ and $\delta > 0$, write

$$\overline{\mathcal{P}}_\delta^h(E) = \sup \left\{ \sum_i h(2r_i) \mid (C(x_i, r_i))_i \text{ is a family of closed balls such that } r_i \leq \delta \text{ and with } x_i \in E \text{ and } d(x_i, x_j) \geq \frac{r_i + r_j}{2} \text{ for } i \neq j \right\}.$$

The h -dimensional prepacking measure $\overline{\mathcal{P}}^h(E)$ of E is now defined by

$$\overline{\mathcal{P}}^h(E) = \inf_{\delta > 0} \overline{\mathcal{P}}_\delta^h(E).$$

Finally, we define the h -dimensional packing measure $\mathcal{P}^h(E)$ of E , as follows

$$\mathcal{P}^h(E) = \inf_{E \subseteq \bigcup_{i=1}^{\infty} E_i} \sum_{i=1}^{\infty} \overline{\mathcal{P}}^h(E_i).$$

As above, we note that if $t > 0$ and h_t denotes the dimension function defined by $h_t(r) = r^t$, then we will follow the traditional convention and write

$$\mathcal{P}^{h_t}(E) = \mathcal{P}^t(E).$$

Finally, the packing dimension $\dim_{\mathcal{P}}(E)$ is defined by

$$\dim_{\mathcal{P}}(E) = \sup\{t \geq 0 \mid \mathcal{P}^t(E) = \infty\}.$$

It is well-known that if $E \subseteq X$, then

$$\begin{aligned} \dim_{\mathcal{H}}(E) &\leq \dim_{\mathcal{P}}(E) \leq \overline{\dim}_{\mathcal{B}}(E), \\ \dim_{\mathcal{H}}(E) &\leq \underline{\dim}_{\mathcal{B}}(E) \leq \overline{\dim}_{\mathcal{B}}(E). \end{aligned}$$

The reader is referred to [2] for an excellent discussion of the Hausdorff dimension, the packing dimension and the box dimensions.

2.3. Hewitt–Stromberg measures

Hewitt–Stromberg measures were introduced by Hewitt & Stromberg in their classical textbook [6, (10.51)], and have subsequently been investigated further by, for example, [4, 5, 14], highlighting their fundamental importance in the study of local properties of fractals and products of fractals. In particular, Edgar’s textbook [1, pp. 32–36], provides an informative and systematic introduction to the Hewitt–Stromberg measures and their importance in the study of local properties of fractals. The measures also appear explicitly in, for example, Pesin’s monograph [10, 5.3], who discusses their important role in the study of dynamical systems and implicitly in Mattila’s text [7]. While Hausdorff and packing measures are defined using coverings and packings by families of sets with diameters *less* than a given positive number δ , say, the Hewitt–Stromberg measures are defined using packings of balls with the *same* diameter δ . For a dimension function h , the Hewitt–Stromberg measures are defined as follows. For a metric space X and $E \subseteq X$, write

$$\begin{aligned} \overline{\mathcal{U}}^h(E) &= \liminf_{r \searrow 0} M_r(E)h(2r). \\ \overline{\mathcal{V}}^h(E) &= \limsup_{r \searrow 0} M_r(E)h(2r). \end{aligned}$$

We now define the lower and upper h -dimensional Hewitt–Stromberg measures, denoted by \mathcal{U}^h and \mathcal{V}^h , respectively, by

$$\begin{aligned} \mathcal{U}^h(E) &= \inf_{E \subseteq \bigcup_{i=1}^{\infty} E_i} \sum_{i=1}^{\infty} \overline{\mathcal{U}}^h(E_i), \\ \mathcal{V}^h(E) &= \inf_{E \subseteq \bigcup_{i=1}^{\infty} E_i} \sum_{i=1}^{\infty} \overline{\mathcal{V}}^h(E_i). \end{aligned}$$

The next result summarises the basic inequalities satisfied by the Hewitt–Stromberg measures, the Hausdorff measure and the packing measure.

Proposition 2.1. *Let h be a dimension function. Then we have*

$$\begin{array}{ccccc} \overline{\mathcal{U}}^h(E) & \leq & \overline{\mathcal{V}}^h(E) & \leq & \overline{\mathcal{P}}^h(E) \\ \vee \downarrow & & \vee \downarrow & & \vee \downarrow \\ \mathcal{H}^h(E) & \leq & \mathcal{U}^h(E) & \leq & \mathcal{V}^h(E) \leq \mathcal{P}^h(E) \end{array}$$

for all metric spaces X and all $E \subseteq X$.

Proof. This follows immediately from the definitions since $N_r(E) \leq M_r(E)$ for all $r > 0$ by [3]; see also [1, pp. 32–36]. \square

2.4. Hewitt–Stromberg measures of typical compact spaces

Our first main result computes the Hewitt–Stromberg measures of a typical compact metric space; this is the content of Theorem 2.2 below.

Theorem 2.2. (Hewitt–Stromberg measures of typical compact spaces) *Let h be a continuous dimension function.*

- (1) *A typical compact metric space $X \in K_{\text{GH}}$ satisfies*

$$\mathcal{U}^h(X) = \overline{\mathcal{U}}^h(X) = 0.$$

- (2) *A typical compact metric space $X \in K_{\text{GH}}$ satisfies*

$$\mathcal{V}^h(U) = \overline{\mathcal{V}}^h(U) = \infty$$

for all non-empty open subsets U of X . In particular, a typical compact metric space $X \in K_{\text{GH}}$ satisfies

$$\mathcal{V}^h(X) = \overline{\mathcal{V}}^h(X) = \infty.$$

The proof of Theorem 2.2 is given in Sect. 3 and Sects. 5–6; Section 3 contains a number of preliminary auxiliary results, and the proofs of the statements in Theorems 2.2.(1) and 2.2.(2) are given in Sects. 5 and 6, respectively.

For brevity write

$$M_{\text{positive}} = \left\{ X \in K_{\text{GH}} \mid 0 < \mathcal{U}^h(X) \right\},$$

$$M_{\text{infinity}} = \left\{ X \in K_{\text{GH}} \mid \mathcal{U}^h(X) = \infty \right\},$$

$$N_{\text{infinity}} = \left\{ X \in K_{\text{GH}} \mid \mathcal{U}^h(U) = \infty \text{ for all non-empty open subsets } U \text{ of } X \right\},$$

and note that

$$N_{\text{infinity}} \subseteq M_{\text{infinity}} \subseteq M_{\text{positive}}.$$

While it follows from Theorem 2.2 that the set M_{positive} is meagre, the set M_{positive} is, nevertheless, dense in K_{GH} . In fact, even the smaller sets N_{infinity} and M_{infinity} are dense in K_{GH} ; this is the content of Theorem 2.3 below.

Theorem 2.3. *Let h be a continuous dimension function. Then the set*

$$\left\{ X \in K_{\text{GH}} \mid \mathcal{U}^h(U) = \infty \text{ for all non-empty open subsets } U \text{ of } X \right\}$$

is dense in K_{GH} .

The proof of Theorem 2.3 is given in Section 4.

We now present several applications of Theorem 2.2. In Section 2.5 we apply Theorem 2.2 to find the Hausdorff and packing measures for a typical compact metric space, and in Section 2.6 we apply the results from Section 2.5 to find the packing dimension (and other dimensions) of a typical compact metric space.

2.5. Hausdorff and packing measures of typical compact spaces

Because of the importance of the Hausdorff measures and the packing measures, the following corollary of Theorem 2.2 seems worthwhile stating separately.

Theorem 2.4. (Hausdorff measures and packing measures of typical compact metric spaces) *Let h be a continuous dimension function.*

- (1) *A typical compact metric space $X \in K_{\text{GH}}$ satisfies*

$$\mathcal{H}^h(X) = 0.$$

- (2) *A typical compact metric space $X \in K_{\text{GH}}$ satisfies*

$$\mathcal{P}^h(U) = \infty$$

for all non-empty open subsets U of X . In particular, a typical compact metric space $X \in K_{\text{GH}}$ satisfies

$$\mathcal{P}^h(X) = \infty.$$

Proof. This result follows immediately from Proposition 2.1 and Theorem 2.2. \square

2.6. Packing dimensions of typical compact spaces

As a further specialization of Theorem 2.4 we obtain the next result about the Hausdorff and packing dimensions of typical compact spaces. While the result in Theorem 2.5.(1) (saying that $\dim_{\text{H}}(X) = 0$ for a typical compact metric space X) has already been obtained by Rouyer [13] (see Theorem A in Section 1), we believe that it is instructive to present a simple proof based on Theorem 2.4.

Theorem 2.5. (Hausdorff dimensions and packing dimensions of typical compact metric spaces) *Let h be a continuous dimension function.*

- (1) [13] A typical compact metric space $X \in K_{\text{GH}}$ satisfies

$$\dim_{\text{H}}(X) = 0.$$

- (2) A typical compact metric space $X \in K_{\text{GH}}$ satisfies

$$\dim_{\text{P}}(U) = \infty$$

for all non-empty open subsets U of X . In particular, a typical compact metric space $X \in K_{\text{GH}}$ satisfies

$$\dim_{\text{P}}(X) = \infty.$$

Proof. (1) Note that

$$\begin{aligned} \bigcap_{t \in \mathbb{Q}^+} \left\{ X \in K_{\text{GH}} \mid \mathcal{H}^t(X) = 0 \right\} &\subseteq \bigcap_{t \in \mathbb{Q}^+} \left\{ X \in K_{\text{GH}} \mid \dim_{\text{H}}(X) \leq t \right\} \\ &\subseteq \left\{ X \in K_{\text{GH}} \mid \dim_{\text{H}}(X) = 0 \right\}. \end{aligned} \quad (2.3)$$

Since it follows from Theorem 2.2 that the set $\{X \in K_{\text{GH}} \mid \mathcal{H}^t(X) = 0\}$ is co-meagre for all $t > 0$, we conclude from (2.3) that the set $\{X \in K_{\text{GH}} \mid \dim_{\text{H}}(X) = 0\}$ is co-meagre.

(2) Note that

$$\begin{aligned} &\bigcap_{t \in \mathbb{Q}^+} \left\{ X \in K_{\text{GH}} \mid \mathcal{P}^t(U) = \infty \text{ for all non-empty open subsets } U \text{ of } X \right\} \\ &\subseteq \bigcap_{t \in \mathbb{Q}^+} \left\{ X \in K_{\text{GH}} \mid \dim_{\text{P}}(U) \geq t \text{ for all non-empty open subsets } U \text{ of } X \right\} \\ &\subseteq \left\{ X \in K_{\text{GH}} \mid \dim_{\text{P}}(U) = \infty \text{ for all non-empty open subsets } U \text{ of } X \right\}. \end{aligned} \quad (2.4)$$

Since it follows from Theorem 2.2 that the set $\{X \in K_{\text{GH}} \mid \mathcal{P}^t(U) = \infty \text{ for all non-empty open subsets } U \text{ of } X\}$ is co-meagre for all $t > 0$, we conclude from (2.4) that the set $\{X \in K_{\text{GH}} \mid \dim_{\text{P}}(U) = \infty \text{ for all non-empty open subsets } U \text{ of } X\}$ is co-meagre. \square

We also obtain the following corollary providing information about the lower box dimension of a typical compact space.

Corollary 2.6. (Lower box dimension of typical compact metric spaces) *Let h be a continuous dimension function such that*

$$\lim_{r \searrow 0} \frac{h(r)}{r^t} = \infty \text{ for all } t > 0 \quad (2.5)$$

(e.g. the dimension function h defined by $h(r) = \frac{1}{\log \frac{1}{r}} k$ for $0 < r < \frac{1}{e}$ and $h(r) = 1$ for $r \geq \frac{1}{e}$ satisfies this condition).

(1) We have

$$\left\{ X \in K_{\text{GH}} \mid \overline{\mathcal{U}}^h(X) = 0 \right\} \subseteq \left\{ X \in K_{\text{GH}} \mid \underline{\dim}_{\text{B}}(X) = 0 \right\}. \quad (2.6)$$

There are continuous dimension functions h satisfying (2.5) such that

$$\left\{ X \in K_{\text{GH}} \mid \overline{\mathcal{U}}^h(X) = 0 \right\} \subsetneq \left\{ X \in K_{\text{GH}} \mid \underline{\dim}_{\text{B}}(X) = 0 \right\}. \quad (2.7)$$

(2) A typical compact space $X \in K_{\text{GH}}$ satisfies

$$\overline{\mathcal{U}}^h(X) = 0,$$

i.e. $\left\{ X \in K_{\text{GH}} \mid \overline{\mathcal{U}}^h(X) = 0 \right\}$ is co-meagre.

(3) [13] A typical compact space $X \in K_{\text{GH}}$ satisfies

$$\underline{\dim}_{\text{B}}(X) = 0,$$

i.e. $\left\{ X \in K_{\text{GH}} \mid \underline{\dim}_{\text{B}}(X) = 0 \right\}$ is co-meagre.

Remark. For brevity write

$$S = \left\{ X \in K_{\text{GH}} \mid \overline{\mathcal{U}}^h(X) = 0 \right\}$$

and

$$T = \left\{ X \in K_{\text{GH}} \mid \underline{\dim}_{\text{B}}(X) = 0 \right\}.$$

The statement in Part (3) of Corollary 2.6 has recently been obtained by Rouyer [13]. However, since Part (1) in Corollary 2.6 shows that S is a subset of T , we deduce that the statement in Part (2) is stronger than Rouyer's result in Part (3). In fact, since Part (1) in Corollary 2.6 also shows that S , in general, is a proper subset of T , we conclude that the statement in Part (2), in general, is *strictly* stronger than Rouyer's result in Part (3).

Proof. (1) The inclusion in (2.6) follows easily from the definitions and the fact that $\lim_{r \searrow 0} \frac{h(r)}{r^t} = \infty$ for all $t > 0$. Next, in order to show (2.7), we must construct a continuous dimension function h satisfying condition (2.5) and a compact metric space X such that $\underline{\dim}_{\text{B}}(X) = 0$ and $\overline{\mathcal{U}}^h(X) > 0$. We construct the space X as follows. For a positive integer n , write $I_n = \{0, 2(n+1) - 1\}$, and for $i \in I_n$ define $S_{n,i} : [0, 1] \rightarrow [0, 1]$ by $S_{n,i}(x) = \frac{1}{2(n+1)}(x + i)$. Next, for $i_1 \in I_1, \dots, i_n \in I_n$, let $I_{i_1 \dots i_n} = S_{1,i_1} S_{2,i_2} \cdots S_{n,i_n}([0, 1])$, and put

$$X_n = \bigcup_{i_1 \in I_1, \dots, i_n \in I_n} I_{i_1 \dots i_n}$$

and

$$X = \bigcap_n X_n ;$$

the set X_n is the union of the 2^n disjoint closed intervals $I_{i_1 \dots i_n}$ each with length equal to $\frac{1}{2^{n(n+1)!}}$, and the sets X_n are constructed inductively as follows: let $X_0 = [0, 1]$ and for $n = 1, 2, \dots$, the set X_n is obtained by deleting the middle $\frac{n}{n+1}$ 'th part of each of the intervals $I_{i_1 \dots i_{n-1}}$ in X_{n-1} .

We first show that $\underline{\dim}_B(X) = 0$. Indeed, if $\frac{1}{2^{n(n+1)!}} < r \leq \frac{1}{2^{(n-1)n!}}$, then X can be covered by 2^n closed intervals with diameter equal to r and so $\underline{\dim}_B(X) \leq \overline{\dim}_B(X) \leq \limsup_{n \rightarrow \infty} \frac{\log 2^n}{-\log \frac{1}{2^{n(n+1)!}}} = 0$.

Next, we construct a continuous dimension function h satisfying (2.5) such that $\mathcal{H}^h(X) > 0$. Indeed, we define h by $h(r) = \frac{1}{\log \frac{1}{r}}$ for $0 < r < \frac{1}{e}$ and $h(r) = 1$ for $r \geq \frac{1}{e}$. It is clear that (2.5) is satisfied. We now show that $\mathcal{H}^h(X) > 0$. Let $\lambda_{i_1 \dots i_n}$ denote the Lebesgue measure restricted to the interval $I_{i_1 \dots i_n}$ and normalised so that $\lambda_{i_1 \dots i_n}(I_{i_1 \dots i_n}) = 1$. Next, define the probability measure μ_n by $\mu_n = \frac{1}{2^n} \sum_{i_1 \in I_1, \dots, i_n \in I_n} \lambda_{i_1 \dots i_n}$. It is not difficult to see that there is a probability measure μ such that μ_n converges weakly to μ . We now show that there is a constant $c > 0$ such that

$$\mu(U) \leq h(\text{diam}(U)) \quad (2.8)$$

for all $U \subseteq [0, 1]$ with $\text{diam}(U) < c$. For a positive integer n , write $r_n = \frac{1}{2^{n(n+1)!}}$. Next, let $U \subseteq [0, 1]$ with $r_{n+1} \leq \text{diam}(U) < r_n$, and note that U can intersect at most one of the intervals $I_{i_1 \dots i_n}$, whence

$$\mu(U) \leq \frac{1}{2^n} = r_{n+1}^{-\frac{\log 2^n}{-\log r_{n+1}}} \leq \text{diam}(U)^{-\frac{\log 2^n}{-\log r_{n+1}}}. \quad (2.9)$$

We now prove the following claim.

Claim 1. There is a positive integer N such that if $n \geq N$ and $r_{n+1} \leq r < r_n$, then

$$r^{-\frac{\log 2^n}{-\log r_{n+1}}} \leq \frac{1}{\log \frac{1}{r}}. \quad (2.10)$$

Proof of Claim 1. Since clearly $\frac{\log \frac{1}{r_n}}{2^{\frac{n}{2}}} = \frac{\log(2^n(n+1)!)}{2^{\frac{n}{2}}} \rightarrow 0$ and $\frac{\log \frac{1}{r_n}}{\log \frac{1}{r_{n+1}}} = \frac{\log(2^n(n+1)!)}{\log(2^{n+1}(n+2)!)} \rightarrow 1$, there is a positive integer N such that $\log \frac{1}{r_n} \leq 2^{\frac{n}{2}}$ for $n \geq N$ and $\log \frac{1}{r_n} \geq \frac{1}{2} \log \frac{1}{r_{n+1}}$ for $n \geq N$. As the function $x \rightarrow \frac{\log x}{x}$ is decreasing for $x \geq e$ and $\log \frac{1}{r_n} \leq \log \frac{1}{r}$, we therefore conclude that $\frac{\log \log \frac{1}{r}}{\log \frac{1}{r}} \leq \frac{\log \log \frac{1}{r_n}}{\log \frac{1}{r_n}} \leq \frac{\log 2^{\frac{n}{2}}}{\frac{1}{2} \log \frac{1}{r_{n+1}}} = \frac{\log 2^n}{\log \frac{1}{r_{n+1}}}$ for all $n \geq N$, and (2.10) follows easily from rearranging this inequality. This completes the proof of Claim 1.

Combining (2.9) and (2.10) we deduce that

$$\mu(U) \leq \text{diam}(U)^{-\frac{\log 2^n}{\log r_n + 1}} \leq \frac{1}{\log \frac{1}{\text{diam}(U)}} = h(\text{diam}(U))$$

provided $\text{diam}(U) < r_N$. This proves inequality (2.8). Finally, it follows from (2.8) and the mass distribution principle that $\mathcal{H}^h(X) \geq 1 > 0$.

(2) This statement follows immediately from Theorem 2.2.

(3) This statement follows immediately from Part (1) and Part (2). \square

3. Proofs of Theorems 2.2 and 2.3: Preliminary results

In this section we collect some basic notation and present several technical auxiliary lemmas that will be used in Sects. 4–6. We first list some useful properties of the covering number $N_r(X)$ and the packing number $M_r(X)$; recall that the covering number $N_r(X)$ and the packing number $M_r(X)$ of a metric space X are defined in (2.1).

Lemma 3.1. (1) *The function $N_r : K_{\text{GH}} \rightarrow \mathbb{R}$ is lower semi-continuous for all $r > 0$.*

(2) *The function $M_r : K_{\text{GH}} \rightarrow \mathbb{R}$ is upper semi-continuous for all $r > 0$.*

(3) *We have $N_r(X) \leq M_r(X) \leq N_{\frac{r}{3}}(X)$ for all $r > 0$ and all $X \in K_{\text{GH}}$.*

Proof. This follows from [13, Lemma 9]; see also [3]. \square

Next, we list some useful properties of the Hewitt–Stromberg measures \mathcal{U}^h and \mathcal{V}^h ; recall that the Hewitt–Stromberg measures \mathcal{U}^h and \mathcal{V}^h are defined in Section 2.3.

Proposition 3.2. *Let h be a continuous dimension function.*

(1) *For all metric spaces X and all $E \subseteq X$, we have $\bar{\mathcal{U}}^h(E) = \bar{\mathcal{U}}^h(\bar{E})$.*

(2) *For all metric spaces X and all $E \subseteq X$, we have $\bar{\mathcal{V}}^h(E) = \bar{\mathcal{V}}^h(\bar{E})$.*

Proof. Let X be a metric space and $E \subseteq X$. It is clear that $\bar{\mathcal{V}}^h(E) \leq \bar{\mathcal{V}}^h(\bar{E})$. We now prove that $\bar{\mathcal{V}}^h(\bar{E}) \leq \bar{\mathcal{V}}^h(E)$. Fix $\varepsilon > 0$. We first prove the following claim.

Claim 1. There are functions $\rho, R : (0, \infty) \rightarrow (0, \infty)$ such that $\frac{1}{2} \leq \rho(r) \leq 1 \leq R(r) \leq 2$ and

$$\begin{aligned} M_r(\bar{E})h(2r) &\leq M_{\rho(r)r}(E)h(2\rho(r)r) + \varepsilon, \\ M_{R(r)r}(\bar{E})h(2R(r)r) &\leq M_r(E)h(2r) + \varepsilon, \end{aligned}$$

for all $r > 0$.

Proof of Claim 1. Let d denote the metric in X . Recall (see Section 2.2) that we use the following notation, namely, if $x \in X$ and $r > 0$, then $C(x, r)$ denotes the closed ball with radius equal to r and centre at x , i.e. $C(x, r) = \{y \in X \mid d(x, y) \leq r\}$.

We now turn towards the proof of Claim 1. Let $r > 0$. Since h is continuous, we can choose a real number $\delta(r)$ with $0 < \delta(r) \leq \frac{1}{2}$ such that

$$h(2r) \leq h(2(1 - \delta(r))r) + \frac{\varepsilon}{M_{\frac{1}{2}r}(E)}. \quad (3.1)$$

It follows from the definition of the packing number $M_r(\overline{E})$ that we can find a family $(C(x_i, r))_{i=1}^{M_r(\overline{E})}$ of closed balls $C(x_i, r)$ in X with $x_i \in \overline{E}$ and $d(x_i, x_j) \geq r$ for $i \neq j$. Since $x_i \in \overline{E}$, there is a point $y_i \in E$ such that $y_i \in B(x_i, \frac{\delta(r)r}{2})$. It therefore follows that $r \leq d(x_i, x_j) \leq d(x_i, y_i) + d(y_i, y_j) + d(y_j, x_j) < \frac{\delta(r)r}{2} + d(y_i, y_j) + \frac{\delta(r)r}{2}$ for $i \neq j$, and so $d(y_i, y_j) \geq r - 2\frac{\delta(r)r}{2} = (1 - \delta(r))r$. Consequently, $(C(y_i, (1 - \delta(r))r))_{i=1}^{M_r(\overline{E})}$ is a family of closed balls with $y_i \in E$ and $d(y_i, y_j) \geq (1 - \delta(r))r$ for $i \neq j$, whence

$$M_r(\overline{E}) \leq M_{(1-\delta(r))r}(E). \quad (3.2)$$

It follows immediately from (3.1) and (3.2) that

$$\begin{aligned} M_r(\overline{E})h(2r) &\leq M_{(1-\delta(r))r}(E) \left(h(2(1 - \delta(r))r) + \frac{\varepsilon}{M_{\frac{1}{2}r}(E)} \right) \\ &= M_{(1-\delta(r))r}(E)h(2(1 - \delta(r))r) + \frac{M_{(1-\delta(r))r}(E)}{M_{\frac{1}{2}r}(E)}\varepsilon. \end{aligned} \quad (3.3)$$

However, since $(1 - \delta(r))r \geq \frac{1}{2}r$, we conclude that $M_{(1-\delta(r))r}(E) \leq M_{\frac{1}{2}r}(E)$, and (3.3) therefore implies that

$$M_r(\overline{E})h(2r) \leq M_{(1-\delta(r))r}(E)h(2(1 - \delta(r))r) + \varepsilon \quad (3.4)$$

for all $r > 0$. Finally, defining $\rho, R : (0, \infty) \rightarrow (0, \infty)$ by $\rho(r) = 1 - \delta(r)$ and $R(r) = \frac{1}{1-\delta(r)}$, the desired conclusion follows immediately from (3.4). This completes the proof of Claim 1.

We can now prove the statement in Proposition 3.2. Since $\frac{1}{2} \leq \rho(r) \leq 1 \leq R(r) \leq 2$, we conclude that

$$\begin{aligned} \inf_{0 < t \leq s} M_t(\overline{E})h(2t) &\leq \inf_{0 < r \leq \frac{1}{2}s} M_{R(r)r}(\overline{E})h(2R(r)r), \\ \sup_{0 < r \leq s} M_{\rho(r)r}(E)h(2\rho(r)r) &\leq \sup_{0 < t \leq s} M_t(E)h(2t), \end{aligned}$$

and it therefore follows from Claim 1 that

$$\begin{aligned}
 \overline{\mathcal{U}}^h(\overline{E}) &= \sup_{s>0} \inf_{0<t\leq s} M_t(\overline{E})h(2t) \\
 &\leq \sup_{s>0} \inf_{0<r\leq \frac{1}{2}s} M_{R(r)r}(\overline{E})h(2R(r)r) \\
 &\leq \sup_{s>0} \inf_{0<r\leq \frac{1}{2}s} M_r(E)h(2r) + \varepsilon \\
 &= \overline{\mathcal{U}}^h(E) + \varepsilon,
 \end{aligned} \tag{3.5}$$

and

$$\begin{aligned}
 \overline{\mathcal{V}}^h(\overline{E}) &= \inf_{s>0} \sup_{0<r\leq s} M_r(\overline{E})h(2r) \\
 &\leq \inf_{s>0} \sup_{0<r\leq s} M_{\rho(r)r}(E)h(2\rho(r)r) + \varepsilon \\
 &\leq \inf_{s>0} \sup_{0<t\leq s} M_t(E)h(2t) + \varepsilon \\
 &\leq \overline{\mathcal{V}}^h(E) + \varepsilon.
 \end{aligned} \tag{3.6}$$

Finally, letting ε tend to 0 in (3.5) and (3.6) gives the desired result. \square

Proposition 3.3. *Let h be a continuous dimension function. Let X be a complete metric space and let C be a compact subset of X . Fix $c \geq 0$.*

- (1) *If $\overline{\mathcal{U}}^h(V \cap C) \geq c$ for all open subsets V of X with $V \cap C \neq \emptyset$, then $\mathcal{U}^h(C) \geq c$.*
- (2) *If $\overline{\mathcal{V}}^h(V \cap C) \geq c$ for all open subsets V of X with $V \cap C \neq \emptyset$, then $\mathcal{V}^h(C) \geq c$.*

Proof. (1) Assume that $\overline{\mathcal{U}}^h(V \cap C) \geq c$ for all open subsets V of X with $V \cap C \neq \emptyset$. We must now show that $\mathcal{U}^h(C) \geq c$. Let $(E_i)_i$ be a countable family of subsets of X with $C \subseteq \cup_i E_i$. We now have $C = \cup_i E_i \subseteq \cup_i \overline{E_i}$, and it therefore follows from Baire's category theorem that there is an index i_0 and an open subset W of X such that $C \cap W \neq \emptyset$ and $C \cap W \subseteq \overline{E_{i_0}}$. We therefore conclude that $\overline{\mathcal{U}}^h(\overline{E_{i_0}}) \geq \overline{\mathcal{U}}^h(C \cap W) \geq c$. It now follows from this and Proposition 3.2 that

$$\begin{aligned}
 \sum_i \overline{\mathcal{U}}^h(E_i) &\geq \overline{\mathcal{U}}^h(E_{i_0}) \\
 &= \overline{\mathcal{U}}^h(\overline{E_{i_0}}) \\
 &\geq c.
 \end{aligned} \tag{3.7}$$

Finally, using (3.7) and taking the infimum over all countable families $(E_i)_i$ of subsets of X with $C \subseteq \cup_i E_i$, shows that $\mathcal{U}^h(E) = \inf_{E \subseteq \cup_{i=1}^\infty E_i} \sum_{i=1}^\infty \overline{\mathcal{U}}^h(E_i) \geq c$.

(2) The proof of this statement is identical to the proof of the statement in Part (1) and is therefore omitted. \square

4. Proof of Theorem 2.3

The purpose of this section is to prove Theorem 2.3. For a dimension function h , we define the set H^h by

$$\begin{aligned}
 H^h = \left\{ X \in K_{\text{GH}} \mid \text{for all } t > 0 \text{ there is a positive integer } N \text{ and} \right. \\
 \quad x_1, \dots, x_N \in X, \\
 \quad C_1, \dots, C_N \subseteq X, \\
 \quad \text{such that} \\
 \quad X = \cup_i B(x_i, t), \\
 \quad C_i \subseteq B(x_i, t) \text{ for all } i, \\
 \quad C_i \in K_{\text{GH}} \text{ for all } i, \\
 \quad \left. \overline{\mathcal{U}}^h(C_i) = \infty \text{ for all } i \right\}. \tag{4.1}
 \end{aligned}$$

Proposition 4.1. *Let h be a dimension function. Then the set H^h is dense in K_{GH} .*

Proof. Let $X \in K_{\text{GH}}$ and let $\rho > 0$. Also, let d_X denote the metric in X . We must now find a compact metric space $Y \in K_{\text{GH}}$ such that $d_{\text{GH}}(X, Y) < \rho$ and $Y \in H^h$. Since X is compact we can choose a finite subset E of X such that

$$d_{\text{H}}(X, E) < \frac{\rho}{2}.$$

Next, define the dimension function $l : (0, \infty) \rightarrow (0, \infty)$ by $l(r) = rh(r)$, and note that it follows from [12, Theorem 36] that there is a compact metric space (Z, d_Z) such that

$$0 < \mathcal{H}^l(Z) < \infty. \tag{4.2}$$

Let μ denote the l -dimensional Hausdorff measure restricted to Z , and write Z_0 for the support of μ , i.e. $Z_0 = \text{supp } \mu$. Next, we fix $z_0 \in Z_0$ and put

$$K = \overline{B(z_0, \frac{\rho}{2})} \cap Z_0.$$

Finally, let

$$Y = E \times K, \tag{4.3}$$

and equip Y with the supremum metric d_Y induced by d_X and d_Z , i.e. $d_Y((x', z'), (x'', z'')) = \max(d_X(x', x''), d_Z(z', z''))$ for $x', x'' \in X$ and $z', z'' \in K$. It is clear that Y is compact, and so $Y \in K_{\text{GH}}$. Below we show that $d_{\text{GH}}(X, Y) < \rho$ and $Y \in H^h$. This is the contents of the two claims below.

Claim 1. $d_{\text{GH}}(X, Y) < \rho$.

Proof of Claim 1. Define $f : E \rightarrow Y$ and $g : Y \rightarrow Y$ by $f(x) = (x, z_0)$ and $g : Y \rightarrow Y$ by $g(x, z) = (x, z)$. It is clear that f and g are isometries and we therefore conclude that $d_{\text{GH}}(E, Y) \leq d_{\text{H}}(f(E), g(Y)) = d_{\text{H}}(E \times \{z_0\}, E \times K) \leq \sup_{z \in K} d_Z(z, z_0) \leq \frac{\rho}{2}$, whence $d_{\text{GH}}(X, Y) \leq d_{\text{GH}}(X, E) + d_{\text{GH}}(E, Y) < d_{\text{H}}(X, E) + \frac{\rho}{2} < \frac{\rho}{2} + \frac{\rho}{2} = \rho$. This completes the proof of Claim 1. \square

Claim 2. $Y \in H^h$.

Proof of Claim 2. Let $t > 0$. It follows from the compactness of K that we can choose finitely many points $z_1, \dots, z_N \in K$ such that $K \subseteq \cup_j B(z_j, t)$. Let $K_j = \overline{B(z_j, \frac{t}{2})} \cap K$ and write $E = \{x_1, \dots, x_M\}$. Finally, put

$$y_{i,j} = (x_i, z_j),$$

$$C_{i,j} = \{x_i\} \times K_j$$

for $i = 1, \dots, M$ and $j = 1, \dots, N$. In order to prove that $Y \in H^h$, it suffices to show that

$$Y = \cup_{i,j} B(y_{i,j}, t), \quad (4.4)$$

$$C_{i,j} \subseteq B(y_{i,j}, t) \quad \text{for all } i, j, \quad (4.5)$$

$$\overline{\mathcal{U}}^h(C_{i,j}) = \infty \quad \text{for all } i, j. \quad (4.6)$$

Below we show that the statements in (4.4)–(4.6) are satisfied.

Indeed, it is clear that $Y = E \times K = \cup_{i,j} (\{x_i\} \times (B(z_j, t) \cap K)) = \cup_{i,j} B(y_{i,j}, t)$; this proves (4.4).

It is also clear that $C_{i,j} \subseteq B(y_{i,j}, t)$ for all i, j ; this proves (4.5).

Finally, we prove (4.6). We first show that

$$0 < \mathcal{H}^l(K_j) < \infty$$

for all j . Indeed, it is clear that $\mathcal{H}^l(K_j) \leq \mathcal{H}^l(Z) < \infty$. Next, we show that $\mathcal{H}^l(K_j) > 0$. Since $z_j \in K = \overline{B(z_0, \frac{\rho}{2})} \cap Z_0$, we conclude that there is a point $\hat{z}_j \in B(z_0, \frac{\rho}{2}) \cap Z_0$ with $d_Z(z_j, \hat{z}_j) < \frac{t}{2}$. Hence, if we write $U_j = B(z_0, \frac{\rho}{2}) \cap B(z_j, \frac{t}{2})$, then $\hat{z}_j \in B(z_0, \frac{\rho}{2}) \cap Z_0$ and $\hat{z}_j \in B(z_j, \frac{t}{2})$, whence $\hat{z}_j \in U_j \cap Z_0$. In particular, we conclude that U_j is an open subset of Z with $U_j \cap Z_0 \neq \emptyset$, and since Z_0 is the support of the l -dimensional Hausdorff measure restricted to Z , we therefore deduce that $\mathcal{H}^l(U_j \cap Z_0) > 0$. Finally, since $K_j = \overline{B(z_j, \frac{t}{2})} \cap K = \overline{B(z_j, \frac{t}{2})} \cap \overline{B(z_0, \frac{\rho}{2})} \cap Z_0 \supseteq B(z_j, \frac{t}{2}) \cap B(z_0, \frac{\rho}{2}) \cap Z_0 = U_j \cap Z_0$, we now infer that $\mathcal{H}^l(K_j) \geq \mathcal{H}^l(U_j \cap Z_0) > 0$.

Since $\mathcal{H}^l(K_j) < \infty$, we can choose $\delta_j > 0$ such that $\mathcal{H}_\delta^l(K_j) \geq \frac{1}{2} \mathcal{H}^l(K_j)$ for all $0 < \delta \leq \delta_j$. This clearly implies that if $0 < \delta \leq \delta_j$ and $(E_i)_i$ is a countable family of subsets of Z with $\text{diam}(E_i) \leq \delta$ and $K_j \subseteq \cup_i E_i$, then

$$\sum_i l(\text{diam}(E_i)) \geq \frac{1}{2} \mathcal{H}^l(K_j). \quad (4.7)$$

Using Lemma 3.1 we deduce that for $\delta > 0$, we have

$$M_\delta(K_j)h(2\delta) \geq N_\delta(K_j)h(2\delta) \geq \frac{1}{2\delta} N_\delta(K_j)l(2\delta). \quad (4.8)$$

Also observe that it follows from the definition of the covering number $N_\delta(K_j)$ that we can find a family $\mathcal{B}_\delta(K_j)$ of $N_\delta(K_j)$ closed balls in Z with centres in K_j and radii equal to δ that covers K_j . In particular, $\text{diam}(C) \leq 2\delta$ for all $C \in \mathcal{B}_\delta(K_j)$, and so

$$N_\delta(K_j)l(2\delta) = \sum_{C \in \mathcal{B}_\delta(K_j)} l(2\delta) \geq \sum_{C \in \mathcal{B}_\delta(K_j)} l(\text{diam}(C)). \quad (4.9)$$

Combining (4.8) and (4.9) now shows that

$$M_\delta(K_j)h(2\delta) \geq \frac{1}{2\delta} N_\delta(K_j)l(2\delta) \geq \frac{1}{2\delta} \sum_{C \in \mathcal{B}_\delta(K_j)} l(\text{diam}(C)). \quad (4.10)$$

However, we conclude from (4.7) that $\sum_{C \in \mathcal{B}_\delta(K_j)} l(\text{diam}(C)) \geq \frac{1}{2} \mathcal{H}^l(K_j)$ for all $0 < \delta \leq \delta_j$, and it therefore follows from (4.10) that

$$M_\delta(K_j)h(2\delta) \geq \frac{1}{2\delta} \sum_{C \in \mathcal{B}_\delta(K_j)} l(\text{diam}(C)) \geq \frac{1}{4\delta} \mathcal{H}^l(K_j)$$

for all $0 < \delta \leq \delta_j$. This clearly implies that

$$\bar{\mathcal{U}}^h(K_j) = \liminf_{\delta \searrow 0} M_\delta(K_j)h(2\delta) \geq \liminf_{\delta \searrow 0} \frac{1}{4\delta} \mathcal{H}^l(K_j) = \infty, \quad (4.11)$$

since $\mathcal{H}^l(K_j) > 0$. Finally, we conclude from (4.11) that $\bar{\mathcal{U}}^h(C_{i,j}) = \bar{\mathcal{U}}^h(\{x_i\} \times K_j) = \bar{\mathcal{U}}^h(K_j) = \infty$. This completes the proof of (4.6).

It follows immediately from (4.4)–(4.6) that $Y \in H^h$. This completes the proof of Claim 2.

Finally, it follows from Claim 1 and Claim 2 that H^h is dense in K_{GH} . \square

Proposition 4.2. *Let h be a continuous dimension function.*

(1) *The set*

$$\left\{ X \in K_{\text{GH}} \mid \bar{\mathcal{U}}^h(U) = \infty \text{ for all open subsets } U \text{ of } X \text{ with } U \neq \emptyset \right\}$$

is dense in K_{GH} .

(2) *The set*

$$\left\{ X \in K_{\text{GH}} \mid \mathcal{U}^h(U) = \infty \text{ for all open subsets } U \text{ of } X \text{ with } U \neq \emptyset \right\}$$

is dense in K_{GH} .

Proof. (1) Using Proposition 4.1, it clearly suffices to show that

$$H^h \subseteq \left\{ X \in K_{\text{GH}} \mid \bar{\mathcal{U}}^h(U) = \infty \text{ for all open subsets } U \text{ of } X \text{ with } U \neq \emptyset \right\}. \quad (4.12)$$

We will now prove (4.12). Let $X \in H^h$. In order to prove (4.12), we must now show that $\bar{\mathcal{U}}^h(U) = \infty$ for all open subsets U of X with $U \neq \emptyset$. We therefore let U be an open subset of X with $U \neq \emptyset$, and proceed to show that

$\overline{\mathcal{U}}^h(U) = \infty$. Since U is non-empty and open there is $x_0 \in U$ and $t_0 > 0$ with $B_X(x_0, t_0) \subseteq U$. Next, since $X \in H^h$, we conclude that there is a positive integer N and

$$\begin{aligned} x_1, \dots, x_N &\in X, \\ C_1, \dots, C_N &\subseteq X, \end{aligned}$$

such that

$$\begin{aligned} X &= \cup_i B(x_i, \frac{t_0}{2}), \\ C_i &\subseteq B(x_i, \frac{t_0}{2}) \quad \text{for all } i, \\ \overline{\mathcal{U}}^h(C_i) &= \infty \quad \text{for all } i. \end{aligned}$$

Since $x_0 \in X = \cup_i B(x_i, \frac{t_0}{2})$, we can choose an index $i_0 \in \{1, \dots, N\}$ with $x_0 \in B(x_{i_0}, \frac{t_0}{2})$, whence $B(x_{i_0}, \frac{t_0}{2}) \subseteq B(x_0, t_0)$, and so $C_{i_0} \subseteq B(x_{i_0}, \frac{t_0}{2}) \subseteq B(x_0, t_0) \subseteq U$. It follows from this that $\overline{\mathcal{U}}^h(U) \geq \overline{\mathcal{U}}^h(C_{i_0}) = \infty$.

(2) Using Part 1, it clearly suffices to prove that

$$\begin{aligned} &\left\{ X \in K_{\text{GH}} \mid \overline{\mathcal{U}}^h(U) = \infty \text{ for all open subsets } U \text{ of } X \text{ with } U \neq \emptyset \right\} \\ &\subseteq \left\{ X \in K_{\text{GH}} \mid \mathcal{U}^h(U) = \infty \text{ for all open subsets } U \text{ of } X \text{ with } U \neq \emptyset \right\}. \end{aligned} \tag{4.13}$$

We will now prove (4.13). Let $X \in K_{\text{GH}}$ and assume that $\overline{\mathcal{U}}^h(U) = \infty$ for all open subsets U of X with $U \neq \emptyset$. In order to prove (4.13), we must now show that $\mathcal{U}^h(U) = \infty$ for all open subsets U of X with $U \neq \emptyset$. We therefore fix an open subset U of X with $U \neq \emptyset$, and proceed to show that $\mathcal{U}^h(U) = \infty$. Since U is non-empty and open there is $x \in U$ and $r > 0$ such that $B_X(x, r) \subseteq U$. In particular, this implies that if we write $C = \overline{B(x, \frac{r}{2})}$, then C is compact and $C \subseteq B(x, r) \subseteq U$. Next, we prove the following claim.

Claim 1. If V is an open subset of X with $V \cap C \neq \emptyset$, then $\overline{\mathcal{U}}^h(V \cap C) = \infty$.

Proof of Claim 1. Let V be an open subset of X with $V \cap C \neq \emptyset$. Choose $y \in V \cap C$. Since $y \in V$ and V is open, we can choose $\varepsilon > 0$ such that $B(y, \varepsilon) \subseteq V$. Next, since $y \in C = \overline{B(x, \frac{r}{2})}$, we can choose $z \in B(x, \frac{r}{2})$ with $z \in B(y, \varepsilon)$. Finally, since $z \in B(x, \frac{r}{2}) \cap B(y, \varepsilon)$, we can find $\delta > 0$ with $B(z, \delta) \subseteq B(x, \frac{r}{2}) \cap B(y, \varepsilon)$, whence $B(z, \delta) \subseteq B(x, \frac{r}{2}) \cap B(y, \varepsilon) \subseteq C \cap V$, and so

$$\overline{\mathcal{U}}^h(B(z, \delta)) \leq \overline{\mathcal{U}}^h(C \cap V). \tag{4.14}$$

However, since the set $B(z, \delta)$ is open and non-empty, it follows from the assumption about X that $\overline{\mathcal{U}}^h(B(z, \delta)) = \infty$, and we therefore conclude from (4.14) that $\overline{\mathcal{U}}^h(C \cap V) = \infty$. This completes the proof of Claim 1.

Finally, it follows immediately from Claim 1 and Proposition 3.3 that $\mathcal{U}^h(C) = \infty$, and since $C \subseteq U$, this implies that $\mathcal{U}^h(U) = \infty$. \square

We can now prove Theorem 2.3.

Proof of Theorem 2.3. Theorem 2.3 follows immediately from Proposition 4.2. (2). \square

5. Proof of Theorem 2.2.(1)

The purpose of this section is to prove Theorem 2.2.(1). For a dimension function h and $r, c > 0$, write

$$L_{r,c}^h = \left\{ X \in K_{\text{GH}} \mid M_r(X) h(2r) < c \right\}.$$

Lemma 5.1. *Let h be a dimension function and $r, c > 0$. Then the set $L_{r,c}^h$ is open in K_{GH} .*

Proof. This follows immediately from Lemma 3.1.

Proposition 5.2. *Let h be a dimension function.*

(1) *For $c \in \mathbb{R}^+$, write*

$$T_c = \left\{ X \in K_{\text{GH}} \mid \overline{\mathcal{U}}^h(X) \leq c \right\}.$$

Then T_c is co-meagre.

(2) *Write*

$$T = \left\{ X \in K_{\text{GH}} \mid \overline{\mathcal{U}}^h(X) = 0 \right\}.$$

Then T is co-meagre.

Proof. (1) It suffices to show that there is a countable family $(G_s)_{s \in \mathbb{Q}^+}$ of open and dense subsets G_s of K_{GH} such that $\bigcap_{s \in \mathbb{Q}^+} G_s \subseteq T_c$. For $s \in \mathbb{Q}^+$, we define the set G_s by

$$G_s = \bigcup_{0 < r < s} L_{r,c}^h.$$

We now prove that the sets G_s are open and dense subsets of K_{GH} such that $\bigcap_{s \in \mathbb{Q}^+} G_s \subseteq T_c$; this is the contents of the three claims below.

Claim 1. G_s is open in K_{GH} .

Proof of Claim 1. This follows immediately from Lemma 5.1. This completes the proof of Claim 1.

Claim 2. G_s is dense in K_{GH} .

Proof of Claim 2. Indeed, it is clear that $\{X \in K_{\text{GH}} \mid X \text{ is finite}\}$ is dense in K_{GH} , and since it is not difficult to see that $\{X \in K_{\text{GH}} \mid X \text{ is finite}\} \subseteq \cup_{0 < r < s} L_{r,c}^h = G_s$, we therefore conclude that G_s is dense in K_{GH} . This completes the proof of Claim 2.

Claim 3. $\cap_{s \in \mathbb{Q}^+} G_s \subseteq T_c$.

Proof of Claim 3. Let $X \in \cap_{s \in \mathbb{Q}^+} G_s$. We must now show that $\bar{U}^h(X) \leq c$. Since $X \in \cap_{s \in \mathbb{Q}^+} G_s \subseteq \cap_n G_{\frac{1}{n}}$, we conclude that for each positive integer n , we can find $r_n < \frac{1}{n}$ such that $X \in L_{r_n,c}^h$, whence $M_{r_n}(X) h(2r_n) < c$. It follows immediately from this that $\bar{U}^h(X) = \liminf_{r \searrow 0} M_r(X) h(2r) \leq \liminf_n M_{r_n}(X) h(2r_n) \leq c$, and so $X \in T_c$. This completes the proof of Claim 3.

(2) This statement follows immediately from Part (1) since clearly $T = \cap_{c \in \mathbb{Q}^+} T_c$.

We can now prove Theorem 2.2.(1).

Proof of Theorem 2.2.(1). Theorem 2.2.(1) follows immediately from Proposition 5.2.(2). \square

6. Proof of Theorem 2.2.(2)

The purpose of this section is to prove Theorem 2.2.(2). We start by introducing the following notation. First, recall that for a positive real number r , the covering number $N_r(X)$ of a metric space X is defined in (2.1). Next, for a dimension function h and $r, t, c > 0$, write

$$\Lambda_{r,c}^h = \left\{ X \in K_{\text{GH}} \mid N_r(X) h(2r) > c \right\},$$

and

$$\begin{aligned} L_{r,t,c}^h = \left\{ X \in K_{\text{GH}} \mid \text{there is a positive integer } N \text{ and} \right. \\ & x_1, \dots, x_N \in X, \\ & C_1, \dots, C_N \subseteq X, \\ & r_1, \dots, r_N \in (0, r), \\ & \text{such that} \\ & X = \cup_i B(x_i, t), \\ & C_i \subseteq B(x_i, t) \text{ for all } i, \\ & \left. C_i \in \Lambda_{r_i,c}^h \text{ for all } i \right\}. \end{aligned}$$

Also recall that for a dimension function h , the set H^h is defined in (4.1).

Lemma 6.1. *Let h be a dimension function and define the dimension function \tilde{h} by $\tilde{h}(r) = h(\frac{r}{3})$ for $r > 0$.*

(1) *For all $X \in K_{\text{GH}}$, we have $\liminf_{r \searrow 0} N_r(X) h(2r) \geq \bar{\mathcal{U}}^{\tilde{h}}(X)$.*

(2) *For all $r, t, c > 0$, we have $H^{\tilde{h}} \subseteq L_{r,t,c}^h$.*

Proof. (1) It follows from Lemma 3.1 that $M_{3r}(X) \leq N_r(X)$ for all $r > 0$, whence $N_r(X) h(2r) \geq M_{3r}(X) h(2r) = M_{3r}(X) \tilde{h}(2 \cdot 3r)$ for all $r > 0$, and so $\liminf_{r \searrow 0} N_r(X) h(2r) \geq \liminf_{r \searrow 0} M_{3r}(X) \tilde{h}(2 \cdot 3r) = \liminf_{r \searrow 0} M_r(C) \tilde{h}(2r) = \bar{\mathcal{U}}^{\tilde{h}}(X)$.

(2) This statement follows immediately from Part (1). \square

Lemma 6.2. *Let h be a dimension function and $r, c > 0$. Then the set $\Lambda_{r,c}^h$ is open in K_{GH} .*

Proof. This follows immediately from Lemma 3.1. \square

Proposition 6.3. *Let h be a dimension function and $r, t, c > 0$. Then the set $L_{r,t,c}^h$ is open in K_{GH} .*

Proof. Let $X \in L_{r,t,c}^h$ and let d_X denote the metric in X . Also, in order to distinguish balls in different metric spaces, we will denote the open ball in X with radius equal to δ and centre at $x \in X$ by $B_X(x, \delta)$, i.e. $B_X(x, \delta) = \{x' \in X \mid d_X(x, x') < \delta\}$.

We must now find $\rho > 0$ such that $B(X, \rho) \subseteq L_{r,t,c}^h$.

Since $X \in L_{r,t,c}^h$, we conclude that there is a positive integer N and

$$\begin{aligned} x_1, \dots, x_N &\in X, \\ C_1, \dots, C_N &\subseteq X, \\ r_1, \dots, r_N &\in (0, r), \end{aligned}$$

such that

$$\begin{aligned} X &= \cup_i B_X(x_i, t), \\ C_i &\subseteq B_X(x_i, t) \quad \text{for all } i, \\ C_i &\in \Lambda_{r_i, c}^h \quad \text{for all } i. \end{aligned}$$

Define $\Phi : X \rightarrow \mathbb{R}$ by $\Phi(x) = \min_i d_X(x, x_i)$ and note that Φ is continuous. Since X is compact, we therefore conclude that there is $x_0 \in X$ such that $\Phi(x_0) = \sup_{x \in X} \Phi(x)$. For brevity write $t_0 = \Phi(x_0) = \sup_{x \in X} \Phi(x)$, and note that since $x_0 \in X = \cup_i B(x_i, t)$, we can find i_0 with $x_0 \in B(x_{i_0}, t)$, whence

$$t_0 = \Phi(x_0) \leq d_X(x, x_{i_0}) < t. \quad (6.1)$$

Also, since C_i is compact and $C_i \subseteq B(x_i, t)$, we conclude that

$$t_i = \inf \left\{ s \mid C_i \subseteq B(x_i, s) \right\} < t. \quad (6.2)$$

For brevity write

$$d_i = t - t_i.$$

Finally, since $C_i \in \Lambda_{r_i,c}^h$ and $\Lambda_{r_i,c}^h$ is open (by Lemma 6.2), we conclude that there is a positive real number $\rho_i > 0$ with

$$B(C_i, \rho_i) \subseteq \Lambda_{r_i,c}^h. \quad (6.3)$$

Now put

$$\rho = \min\left(\frac{\rho_1}{2}, \dots, \frac{\rho_N}{2}, \frac{t-t_0}{2}, \frac{d_1}{16}, \dots, \frac{d_N}{16}\right).$$

It follows from (6.1) and (6.2) that $\rho > 0$. We will now prove that

$$B(X, \rho) \subseteq L_{r,t,c}^h. \quad (6.4)$$

Let $Y \in B(X, \rho)$ and let d_Y denote the metric in Y . Since $d_{\text{GH}}(X, Y) < \rho$, it follows that we may assume that there is a complete metric space (Z, d_Z) with $X, Y \subseteq Z$ and $d_{\text{H}}(X, Y) < \rho$ such that $d_X(x', x'') = d_Z(x', x'')$ for all $x', x'' \in X$, and $d_Y(y', y'') = d_Z(y', y'')$ for all $y', y'' \in Y$. Below we use the following notation allowing us to distinguish balls in Y and balls in Z . Namely, we will denote the open ball in Y with radius equal to δ and centre at $y \in Y$ by $B_Y(y, \delta)$, i.e. $B_Y(y, \delta) = \{y' \in Y \mid d_Y(y, y') < \delta\}$, and we will denote the open ball in Z with radius equal to δ and centre at $z \in Z$ by $B_Z(z, \delta)$, i.e. $B_Z(z, \delta) = \{z' \in X \mid d_Z(z, z') < \delta\}$.

We must now show that $Y \in L_{r,t,c}^h$. Since $d_{\text{H}}(X, Y) < \rho$, we conclude that for each i , there is a point $y_i \in Y$ with $d_Z(x_i, y_i) < \rho$. Next, put

$$K_i = \left\{ y \in Y \mid \text{dist}(y, C_i) \leq \rho \right\}.$$

It is clear that

$$\begin{aligned} y_1, \dots, y_N &\in Y, \\ K_1, \dots, K_N &\subseteq Y, \\ r_1, \dots, r_N &\in (0, r). \end{aligned}$$

In order to prove that $Y \in L_{r,t,c}^h$, it suffices to show that

$$Y = \cup_i B_Y(y_i, t), \quad (6.5)$$

$$K_i \subseteq B_Y(y_i, t) \quad \text{for all } i, \quad (6.6)$$

$$K_i \in \Lambda_{r_i,c}^h \quad \text{for all } i. \quad (6.7)$$

The proofs of (6.5)–(6.7) are the contents of the three claims below.

Claim 1. $Y = \cup_i B_Y(y_i, t)$.

Proof of Claim 1. It is clear that $\cup_i B_Y(y_i, t) \subseteq Y$. In order to prove the reverse inclusion, we let $y \in Y$. Since $d_{\text{H}}(X, Y) < \rho$, we conclude that there is a point $x \in X$ with $d_Z(x, y) < \rho$. Also, since $\min_i d_X(x, x_i) = \Phi(x) \leq t_0$, we deduce that there is an index j with $d_X(x, x_j) \leq t_0$. Finally, it follows

from the definition of y_j that $d_Z(x_j, y_j) < \rho$. Hence $d_Y(y, y_j) = d_Z(y, y_j) \leq d_Z(y, x) + d_Z(x, x_j) + d_Z(x_j, y_j) = d_Z(y, x) + d_X(x, x_j) + d_Z(x_j, y_j) < \rho + t_0 + \rho = 2\rho + t_0 \leq t$, and so $y \in B_Y(y_j, t) \subseteq \cup_i B_Y(y_i, t)$. This completes the proof of Claim 1.

Claim 2. $K_i \subseteq B(y_i, t)$ for all i .

Proof of Claim 2. Since $C_i \subseteq B_X(x_i, t)$, it follows from the definition of the numbers $t_i = \inf\{s \mid C_i \subseteq B(x_i, s)\}$ and $d_i = t - t_i$, that

$$C_i \subseteq B_X(x_i, t - \frac{d_i}{2}). \quad (6.8)$$

Next, since $d_Z(x_i, y_i) < \rho \leq \frac{d_i}{16} \leq \frac{d_i}{4}$, it follows that

$$\begin{aligned} B_X(x_i, t - \frac{d_i}{2}) &\subseteq B_Z(x_i, t - \frac{d_i}{2}) \\ &\subseteq B_Z(y_i, t - \frac{d_i}{4}). \end{aligned} \quad (6.9)$$

Finally, combining (6.8) and (6.9) shows that

$$C_i \subseteq B_Z(y_i, t - \frac{d_i}{4}). \quad (6.10)$$

We can now prove that $K_i \subseteq B_Y(y_i, t)$. Let $y \in K_i$. Since $y \in K_i$, we have $\text{dist}(y, C_i) \leq \rho \leq \frac{d_i}{16} < \frac{d_i}{8}$, and it therefore follows that there is $x \in C_i$ with $d_Z(x, y) \leq \frac{d_i}{8}$. Also, we deduce from (6.10) that $x \in C_i \subseteq B_Z(y_i, t - \frac{d_i}{4})$, whence $d_Z(x, y_i) \leq t - \frac{d_i}{4}$. Combining the previous inequalities we have $d_Y(y, y_i) = d_Z(y, y_i) \leq d_Z(y, x) + d_Z(x, y_i) \leq \frac{d_i}{8} + t - \frac{d_i}{4} < t$, and so $y \in B_Y(y_i, t)$. This completes the proof of Claim 2.

Claim 3. $K_i \in \Lambda_{r_i, c}^h$ for all i .

Proof of Claim 3. It is clear that K_i is a closed subset of Y and so $K_i \in K_{\text{GH}}$.

We now prove that

$$\sup_{x \in C_i} \text{dist}(x, K_i) \leq \rho. \quad (6.11)$$

Indeed, let $x \in C_i$. Since $d_H(X, Y) < \rho$, we conclude that there is $y \in Y$ such that $d_Z(x, y) < \rho$. In particular, since $x \in C_i$, this shows that $\text{dist}(y, C_i) \leq d_Z(y, x) \leq \rho$, and so $y \in K_i$. We deduce from this that $\text{dist}(x, K_i) \leq d_Z(x, y) \leq \rho$. Finally, taking the supremum over all $x \in C_i$ shows that $\sup_{x \in C_i} \text{dist}(x, K_i) \leq \rho$. This completes the proof of (6.11).

Next, we prove that

$$\sup_{y \in K_i} \text{dist}(y, C_i) \leq \rho. \quad (6.12)$$

Indeed, let $y \in K_i$. Since $y \in K_i$, it follows from the definition of K_i that there is $x \in C_i$ such that $d_Z(y, x) \leq \rho$, and so $\text{dist}(y, C_i) \leq d_Z(y, x) \leq \rho$. Finally, taking the supremum over all $y \in K_i$ shows that $\sup_{y \in K_i} \text{dist}(y, C_i) \leq \rho$. This completes the proof of (6.12).

Combining (6.11) and (6.12), we immediately conclude that $d_H(C_i, K_i) = \max(\sup_{x \in C_i} \text{dist}(x, K_i), \sup_{y \in K_i} \text{dist}(y, C_i)) \leq \rho \leq \frac{\rho_i}{2} < \rho_i$, whence $K_i \in B(C_i, \rho_i) \subseteq \Lambda_{r_i, c}^h$. This completes the proof of Claim 3.

It follows immediately from Claim 1–Claim 3 that $Y \in L_{r, t, c}^h$. \square

Proposition 6.4. *Let h be a continuous dimension function.*

(1) *For $c \in \mathbb{R}^+$, write*

$$T_c = \left\{ X \in K_{\text{GH}} \mid \overline{\mathcal{V}}^h(U) \geq c \text{ for all open subsets } U \text{ of } X \text{ with } U \neq \emptyset \right\}.$$

Then T_c is co-meagre.

(2) *Write*

$$T = \left\{ X \in K_{\text{GH}} \mid \overline{\mathcal{V}}^h(U) = \infty \text{ for all open subsets } U \text{ of } X \text{ with } U \neq \emptyset \right\}.$$

Then T is co-meagre.

(3) *Write*

$$S = \left\{ X \in K_{\text{GH}} \mid \mathcal{V}^h(U) = \infty \text{ for all open subsets } U \text{ of } X \text{ with } U \neq \emptyset \right\}.$$

Then S is co-meagre.

Proof. (1) It suffices to show that there is a countable family $(G_{s,t})_{s,t \in \mathbb{Q}^+}$ of open and dense subsets $G_{s,t}$ of K_{GH} such that $\bigcap_{s,t \in \mathbb{Q}^+} G_{s,t} \subseteq T_c$. For $s, t \in \mathbb{Q}^+$, we define the set $G_{s,t}$ by

$$G_{s,t} = \bigcup_{0 < r < s} L_{r, t, c}^h.$$

We now prove that the sets $G_{s,t}$ are open and dense subsets of K_{GH} such that $\bigcap_{s,t \in \mathbb{Q}^+} G_{s,t} \subseteq T_c$; this is the contents of the three claims below.

Claim 1. $G_{s,t}$ is open in K_{GH} .

Proof of Claim 1. This follows immediately from Proposition 6.3. This completes the proof of Claim 1.

Claim 2. $G_{s,t}$ is dense in K_{GH} .

Proof of Claim 2. Let \tilde{h} denote the dimension function defined by $\tilde{h}(r) = h(\frac{r}{3})$ for $r > 0$, and note that it follows from Proposition 4.1 that $H^{\tilde{h}}$ is dense in K_{GH} . Since it also follows from Lemma 6.1 that $H^{\tilde{h}} \subseteq \bigcup_{0 < r < s} L_{r, t, c}^h = G_{s,t}$, we therefore conclude that $G_{s,t}$ is dense in K_{GH} . This completes the proof of Claim 2.

Claim 3. $\bigcap_{s,t \in \mathbb{Q}^+} G_{s,t} \subseteq T_c$.

Proof of Claim 3. Let $X \in \cap_{s,t \in \mathbb{Q}^+} G_{s,t}$. We must now show that if U is an open subset of X with $U \neq \emptyset$, then $\bar{\mathcal{V}}^h(U) \geq c$. We therefore let U be an open subset of X with $U \neq \emptyset$, and proceed to show that $\bar{\mathcal{V}}^h(U) \geq c$. Since U is non-empty and open there is $x_0 \in U$ and $t_0 > 0$ with $B(x_0, t_0) \subseteq U$. Next, since $X \in \cap_{s,t \in \mathbb{Q}^+} G_{s,t} \subseteq \cap_n G_{\frac{1}{n}, \frac{t_0}{2}}$, we conclude that for each positive integer n , we can find a positive real number r_n with $r_n < \frac{1}{n}$ such that $X \in L_{r_n, \frac{t_0}{2}, c}^h$. In particular, this implies that there is a positive integer N_n and

$$\begin{aligned} x_{n,1}, \dots, x_{n,N_n} &\in X, \\ C_{n,1}, \dots, C_{n,N_n} &\subseteq X, \\ r_{n,1}, \dots, r_{n,N_n} &\in (0, r_n), \end{aligned}$$

such that

$$\begin{aligned} X &= \cup_i B(x_{n,i}, \frac{t_0}{2}), \\ C_{n,i} &\subseteq B(x_{n,i}, \frac{t_0}{2}) \quad \text{for all } i, \\ C_{n,i} &\in \Lambda_{r_{n,i}, c}^h \quad \text{for all } i. \end{aligned}$$

Since $x_0 \in X = \cup_i B(x_{n,i}, \frac{t_0}{2})$, we can choose an index $i_n \in \{1, \dots, N_n\}$ such that $x_0 \in B(x_{n,i_n}, \frac{t_0}{2})$, whence $B(x_{n,i_n}, \frac{t_0}{2}) \subseteq B(x_0, t_0)$, and so $C_{n,i_n} \subseteq B(x_{n,i_n}, \frac{t_0}{2}) \subseteq B(x_0, t_0) \subseteq U$. We conclude from this and Lemma 3.2 together with the fact that $C_{n,i_n} \in \Lambda_{r_{n,i_n}, c}^h$, that $M_{r_{n,i_n}}(U) h(2r_{n,i_n}) \geq N_{r_{n,i_n}}(U) h(2r_{n,i_n}) \geq N_{r_{n,i_n}}(C_{n,i_n}) h(2r_{n,i_n}) > c$. Finally, since $r_{n,i_n} < r_n < \frac{1}{n}$ and so $r_{n,i_n} \rightarrow 0$, we deduce from the previous inequality that $\bar{\mathcal{V}}^h(U) = \limsup_{r \searrow 0} M_r(U) h(2r) \geq \limsup_n M_{r_{n,i_n}}(U) h(2r_{n,i_n}) \geq c$. This completes the proof of Claim 3.

(2) This statement follows immediately from Part (1) since clearly $T = \cap_{c \in \mathbb{Q}^+} T_c$.

(3) Using Part (2), it clearly suffices to prove that

$$T \subseteq S. \tag{6.13}$$

To the end, let $X \in T$. We must now show that if U is an open subset of X with $U \neq \emptyset$, then $\mathcal{V}^h(U) = \infty$. We therefore let U be an open subset of X with $U \neq \emptyset$, and proceed to show that $\mathcal{V}^h(U) = \infty$. Since U is non-empty and open there is $x \in U$ and $r > 0$ such that $B(x, r) \subseteq U$. In particular, this implies that if we write $C = \overline{B(x, \frac{r}{2})}$, then C is compact and $C \subseteq B(x, r) \subseteq U$. Next, we prove the following claim.

Claim 4. If V is an open subset of X with $V \cap C \neq \emptyset$, then $\bar{\mathcal{V}}^h(V \cap C) = \infty$.

Proof of Claim 4. Let V be an open subset of X with $V \cap C \neq \emptyset$. We must now show that $\bar{\mathcal{V}}^h(V \cap C) = \infty$. As $V \cap C \neq \emptyset$, it is possible to choose $y \in V \cap C$. Since $y \in V$ and V is open, we can choose $\varepsilon > 0$ such that $B(y, \varepsilon) \subseteq V$. Next, since $y \in C = \overline{B(x, \frac{r}{2})}$, we choose $z \in B(x, \frac{r}{2})$ such that $z \in B(y, \varepsilon)$. Finally,

since $z \in B(x, \frac{r}{2}) \cap B(y, \varepsilon)$, we can find $\delta > 0$ with $B(z, \delta) \subseteq B(x, \frac{r}{2}) \cap B(y, \varepsilon)$, whence $B(z, \delta) \subseteq B(x, \frac{r}{2}) \cap B(y, \varepsilon) \subseteq C \cap V$, and so

$$\overline{\mathcal{V}}^h(B(z, \delta)) \leq \overline{\mathcal{V}}^h(C \cap V). \quad (6.14)$$

However, since $B(z, \delta)$ is open and non-empty and $X \in T$, it follows that $\overline{\mathcal{V}}^h(B(z, \delta)) = \infty$, and we therefore conclude from (6.14) that $\overline{\mathcal{V}}^h(C \cap V) = \infty$. This completes the proof of Claim 4.

Finally, it follows immediately from Claim 4 and Proposition 3.3 that $\mathcal{V}^h(C) = \infty$, and since $C \subseteq U$, this implies that $\mathcal{V}^h(U) = \infty$.

We can now prove Theorem 2.2.(2).

Proof of Theorem 2.2.(2). Theorem 2.2.(2) follows immediately from Proposition 6.4.(3). \square

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